ATKINSON SCIENCE

THEORY GUIDE

Equations of Fluid Flow

Scalar Conservation Equation in Cartesian and Cylindrical Coordinates

Keith Atkinson

10 October 2020

Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

Contents

Figures

1 Scalar conservation equation in Cartesian coordinates

1.1 Control volume analysis

The rate of change of any scalar property Φ in a control volume is equal to the net rate at which the scalar property enters the control volume by convection, plus the net rate at which the scalar property enters the control volume by diffusion, plus the rate of creation or destruction of the scalar property by an external source. The processes are set out in (1.1).

$$\boxed{\begin{array}{l}\text{Rate of increase}\\ \text{of } \Phi \text{ in CV}\end{array}} = \boxed{\begin{array}{l}\text{Net rate of flow}\\ \text{of } \Phi \text{ into CV by}\\ \text{convection}\end{array}} + \boxed{\begin{array}{l}\text{Net rate of flow}\\ \text{of } \Phi \text{ into CV by}\\ \text{diffusion}\end{array}}$$

$$+ \boxed{\begin{array}{l}\text{Rate of creation}\\ \text{or destruction of}\\ \Phi \text{ in CV}\end{array}} \tag{1.1}$$

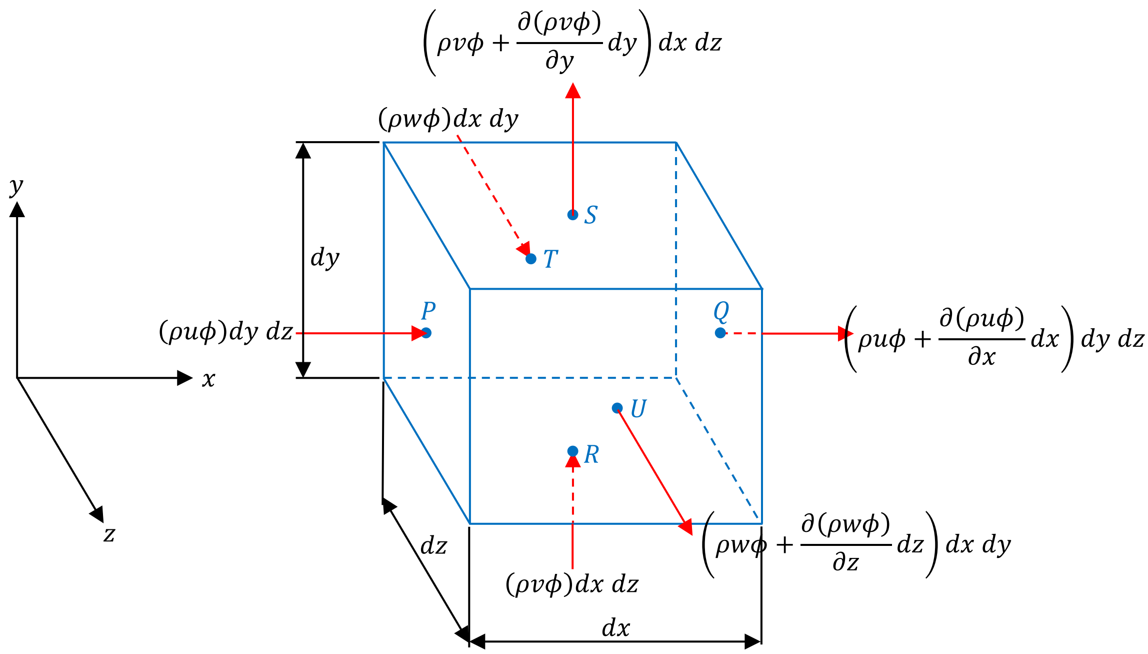
The *mass fraction* of a scalar property in a mixture is the mass of the property per unit mass of mixture. We shall denote the mass fraction of Φ by ϕ [kg Φ kg^{-1} mixture].

1.2 Transient and convection terms

Figure 1 shows an infinitesimal rectangular control volume (CV) through which a fluid mixture containing a scalar Φ is flowing. The mass of scalar Φ in the CV is equal to the mass fraction ϕ [kg Φ kg^{-1}] times the mass of mixture in the CV, $\rho\, dx\, dy\, dz$ [kg]; that is, $\rho\phi\, dx\, dy\, dz$ [kg Φ]. Note that kg Φ is taken to mean kg of scalar Φ and kg alone is taken to mean kg of mixture. The rate of increase of scalar Φ with time, the left-hand term in (1.1), is therefore

$$\frac{\partial(\rho\phi)}{\partial t} dx\, dy\, dz \quad [\text{kg}\,\Phi\,\text{s}^{-1}] \quad (1.2)$$

Figure 1 Infinitesimal control volume for Cartesian coordinates



The scalar may enter or leave through any of the faces P to U in Figure 1, transported by the mass flow of mixture through the faces. The rate of flow of scalar Φ through the face perpendicular to the x direction whose centre is P is ϕ [kg Φ kg^{-1}] times the mass flow of mixture through the face, $\rho u \, dy \, dz$ [kg s^{-1}]; that is,

$$\rho u \phi \, dy \, dz \quad [\text{kg } \Phi \text{ s}^{-1}]$$

The rate of flow of scalar Φ through the opposite face whose centre is Q is

$$\left(\rho u \phi + \frac{\partial(\rho u \phi)}{\partial x} dx \right) dy \, dz \quad [\text{kg } \Phi \text{ s}^{-1}]$$

and so the net rate of flow of scalar Φ *out* of the CV through the faces with centres P and Q is

$$\left(\rho u \phi + \frac{\partial(\rho u \phi)}{\partial x} dx \right) dy \, dz - \rho u \phi \, dy \, dz$$

$$= \frac{\partial(\rho u \phi)}{\partial x} dx \, dy \, dz \quad [\text{kg } \Phi \text{ s}^{-1}]$$

The rate of flow of scalar Φ through the face perpendicular to the y direction whose centre is R is ϕ [kg Φ kg^{-1}] times the mass flow of mixture through the face, $\rho v \, dx \, dz$ [kg s^{-1}]; that is,

$$\rho v \phi \, dx \, dz \quad [\text{kg } \Phi \text{ s}^{-1}]$$

The rate of flow of scalar Φ through the opposite face whose centre is S is

$$\left(\rho v \phi + \frac{\partial(\rho v \phi)}{\partial y} dy \right) dx \, dz \quad [\text{kg } \Phi \text{ s}^{-1}]$$

and so the net rate of flow out of the CV through the faces with centres R and S is

$$\left(\rho v \phi + \frac{\partial(\rho v \phi)}{\partial y} dy \right) dx \, dz - \rho v \phi \, dx \, dz$$

$$= \frac{\partial(\rho v \phi)}{\partial y} dx \, dy \, dz \quad [\text{kg } \Phi \text{ s}^{-1}]$$

Similarly, the net rate of flow out of the CV through the faces normal to the z axis with centres T and U is

$$\frac{\partial(\rho w \phi)}{\partial z} dx \, dy \, dz \quad [\text{kg } \Phi \text{ s}^{-1}]$$

Adding together the terms for the three pairs of faces, the sum of the net rates of outflow of scalar Φ is

$$\left[\frac{\partial(\rho u\phi)}{\partial x}+\frac{\partial(\rho v\phi)}{\partial y}+\frac{\partial(\rho w\phi)}{\partial z}\right]dx\,dy\,dz \qquad [\mathrm{kg}\,\Phi\,\mathrm{s}^{-1}]$$

The net rate of inflow of Φ by convection, the second term in (1.1), is therefore

$$-\left[\frac{\partial(\rho u\phi)}{\partial x}+\frac{\partial(\rho v\phi)}{\partial y}+\frac{\partial(\rho w\phi)}{\partial z}\right]dx\,dy\,dz \qquad [\mathrm{kg}\,\Phi\,\mathrm{s}^{-1}] \qquad (1.3)$$

1.3 Diffusion terms

The third term on the right of (1.1) represents the net flow rate of Φ into the CV by diffusion. We shall denote the diffusion flux per unit area by $\boldsymbol{q}$ [kg Φ m^{-2} s^{-1}]. $\boldsymbol{q}$ has components q_x, q_y, and q_z [kg Φ m^{-2} s^{-1}] in the x, y and z coordinate directions, respectively. Diffusion is considered positive if it is in the positive coordinate direction.

Referring to Figure 2, the rate of diffusion of Φ through the face perpendicular to the x direction whose centre is P is q_x [kg Φ m^{-2} s^{-1}] times the area of the face, $dy\,dz$ [m^2]; that is,

$$q_x\,dy\,dz \qquad [\text{kg}\,\Phi\,\text{s}^{-1}]$$

The rate of diffusion of Φ through the opposite face whose centre is Q is

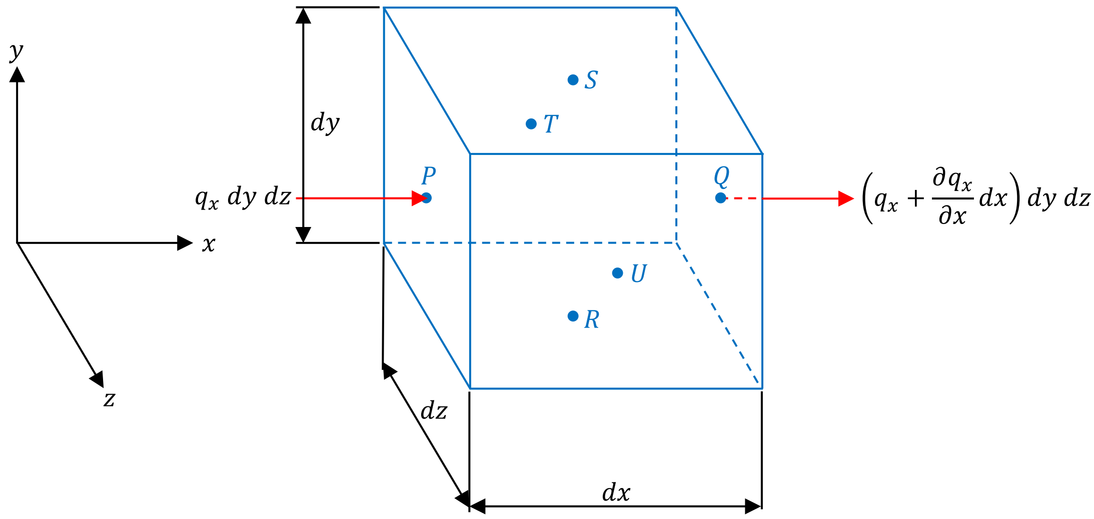
$$\left(q_x + \frac{\partial q_x}{\partial x}dx\right)dy\,dz \qquad [\text{kg}\,\Phi\,\text{s}^{-1}]$$

and so the net rate of diffusion *out* of the CV through the faces with centres P and Q is

$$\left(q_x + \frac{\partial q_x}{\partial x}dx\right)dy\,dz - q_x\,dy\,dz$$

$$= \frac{\partial q_x}{\partial x}dx\,dy\,dz \qquad [\text{kg}\,\Phi\,\text{s}^{-1}]$$

Figure 2 Diffusion of Φ in the x direction



Similarly, the net rate of diffusion of Φ out of the CV through the faces normal to the y axis with centres R and S is

$$\frac{\partial q_y}{\partial y} dx\, dy\, dz \quad [\mathrm{kg}\, \Phi\, \mathrm{s}^{-1}]$$

and the net rate of diffusion of Φ out of the CV through the faces normal to the z axis with centres T and U is

$$\frac{\partial q_z}{\partial z} dx\, dy\, dz \quad [\mathrm{kg}\, \Phi\, \mathrm{s}^{-1}]$$

The rate of diffusion of Φ *into* the CV is therefore

$$-\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] dx\, dy\, dz \quad [\mathrm{kg}\, \Phi\, \mathrm{s}^{-1}]$$

The scalar mass fluxes q_x, q_y, and q_z [kg Φ m^{-2} s^{-1}] in this equation are related to the scalar property gradients by *Fick's law of diffusion*:

$$q_x = -\rho D \frac{\partial \phi}{\partial x}$$

$$q_y = -\rho D \frac{\partial \phi}{\partial y}$$

$$q_z = -\rho D \frac{\partial \phi}{\partial z}$$

where D [m^2 s^{-1}] is a diffusion coefficient. The net rate of diffusion of Φ into the CV, the third term in (1.1), is therefore

$$\left[\frac{\partial}{\partial x}\left(\rho D \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho D \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\rho D \frac{\partial \phi}{\partial z}\right)\right] dx\, dy\, dz \quad [\mathrm{kg}\, \Phi\, \mathrm{s}^{-1}] \quad (1.4)$$

1.4 Source term

The scalar Φ may be created or destroyed as the mixture flows through the CV. If S_Φ [kg Φ kg^{-1} s^{-1}] is the rate at which Φ is created or destroyed per unit mass of mixture then the rate of creation or destruction of Φ in the CV, the fourth term in (1.1), is

$$\rho S_\Phi dx\, dy\, dz \qquad [\mathrm{kg}\ \Phi\ \mathrm{s}^{-1}] \qquad (1.5)$$

1.5 Scalar conservation equation

Substituting the terms (1.2), (1.3), (1.4) and (1.5) into (1.1) and dividing by $dx\ dy\ dz$ gives the conservation equation for the scalar property Φ in Cartesian coordinates:

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{\partial(\rho u\phi)}{\partial x}+\frac{\partial(\rho v\phi)}{\partial y}+\frac{\partial(\rho w\phi)}{\partial z}$$

$$=\frac{\partial}{\partial x}\left(\rho D\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\rho D\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\rho D\frac{\partial\phi}{\partial z}\right)+\rho S_\Phi \qquad [\ \mathrm{kg}\ \Phi\ \mathrm{m}^{-3}\ \mathrm{s}^{-1}] \qquad (1.6)$$

2 Scalar conservation equation in cylindrical coordinates

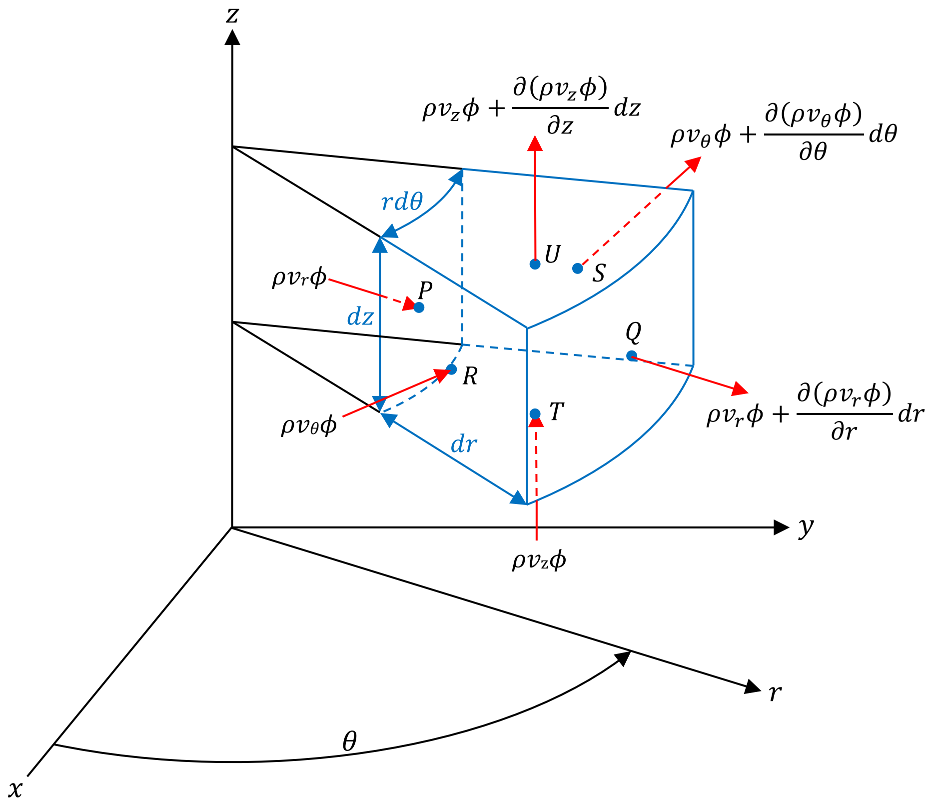
2.1 Control volume analysis

We can derive the scalar conservation equation in cylindrical coordinates based on the concept of an infinitesimal control volume, just as we did with the scalar conservation equation in Cartesian coordinates. This time we consider the properties of the flow into and out of the infinitesimal annular control volume (CV) shown in Figure 3. The lengths of the sides dr, $rd\theta$ and dz are small enough for us to be able to neglect quantities of order dr^2, $r^2d\theta^2$ or dz^2. Recalling (1.1), the conservation principle for any scalar property Φ can be written

Rate of increase of Φ in CV	=	Net rate of flow of Φ into CV by convection	+	Net rate of flow of Φ into CV by diffusion
+	Rate of creation or destruction of Φ by an external source			

The *mass fraction* of a scalar property in a mixture is the mass of the property per unit mass of mixture. We shall denote the mass fraction of Φ by ϕ [kg Φ kg^{-1} mixture].

Figure 3 Infinitesimal control volume for cylindrical coordinates



2.2 Transient and convection terms

Figure 3 shows an infinitesimal annular control volume (CV) through which a fluid mixture containing a scalar Φ is flowing. The mass of scalar Φ in the CV is equal to the mass fraction ϕ [kg Φ kg^{-1}] times the mass of mixture in the CV, $\rho\ dr\ rd\theta\ dz$ [kg]; that is, $\rho\phi\ dx\ dy\ dz$ [kg Φ]. Note that kg Φ is taken to mean kg of scalar Φ and kg alone is taken to mean kg of mixture. The rate of increase of scalar Φ with time, the left-hand term in (1.1), is therefore

$$\frac{\partial(\rho\phi)}{\partial t} dr\ rd\theta\ dz \qquad [\text{kg}\ \Phi\ \text{s}^{-1}] \qquad (2.1)$$

The scalar may enter or leave through any of the faces P to U in Figure 3, transported by the mass flow of mixture through the faces. The rate of flow of scalar Φ through the face perpendicular to the r direction whose centre is P is ϕ [kg Φ kg^{-1}] times the mass flow of mixture through the face, $\rho\ v_r\ rd\theta\ dz$; that is,

$$\rho v_r \phi\ rd\theta\ dz \qquad [\text{kg}\ \Phi\ \text{s}^{-1}]$$

The rate of flow of scalar Φ through the opposite face whose centre is Q is

$$\left(\rho u\phi + \frac{\partial(\rho u\phi)}{\partial r} dr\right)(r + dr)d\theta\ dz \qquad [\text{kg}\ \Phi\ \text{s}^{-1}]$$

and so the net rate of flow of scalar Φ *out* of the CV through the faces with centres P and Q is

$$\left(\rho v_r\phi + \frac{\partial(\rho v_r\phi)}{\partial r} dr\right)(r + dr)d\theta\ dz - (\rho v_r\phi)\ rd\theta\ dz =$$

$$\frac{\partial(\rho v_r\phi)}{\partial r} dr\ rd\theta\ dz + \rho v_r\phi\ dr\ d\theta\ dz + \frac{\partial(\rho v_r\phi)}{\partial r} dr^2\ d\theta\ dz$$

We can neglect the term in dr^2, so the net rate of flow of scalar Φ out of the CV through the faces with centres P and Q is

$$\frac{\partial(\rho v_r\phi)}{\partial r} dr\ rd\theta\ dz + \rho v_r\phi\ dr\ d\theta\ dz$$

The rate of flow of scalar Φ through the face perpendicular to the θ direction whose centre is R is ϕ [kg Φ kg^{-1}] times the mass flow of mixture through the face, face, $\rho\ v_\theta\ dr\ dz$; that is,

$$(\rho v_\theta\phi)\ dr\ dz$$

The corresponding rate of flow of scalar Φ out of the face with centre S is

$$\left(\rho v_\theta\phi + \frac{\partial(\rho v_\theta\phi)}{\partial\theta} d\theta\right) dr\ dz$$

so the net rate of flow of scalar Φ out of the CV through the faces with centres R and S is

$$\frac{\partial(\rho v_\theta\phi)}{\partial\theta} dr\ d\theta\ dz$$

The rate of flow of scalar Φ through the face perpendicular to the z direction with centre T is ϕ [J kg^{-1}] times the mass flow through the face. The area of the face is

$$dr\ (r + ½dr)d\theta$$

so the rate of flow of mass through the face is

$$(\rho v_z)\ dr\ (r + ½dr)d\theta$$

and the rate of flow of scalar Φ through the face is

$$(\rho v_z \phi)\ dr\ (r + ½dr)d\theta$$

The corresponding rate of flow of scalar Φ out of the face with centre U is

$$\left(\rho v_z \phi + \frac{\partial(\rho v_z \phi)}{\partial z} dz\right) dr\ (r + ½dr)d\theta$$

so the net rate of flow of scalar Φ out of the CV through the faces with centres T and U is

$$\frac{\partial(\rho v_z \phi)}{\partial z} dr\ (r + ½dr)d\theta\ dz = \frac{\partial(\rho v_z \phi)}{\partial z} dr\ rd\theta\ dz + \frac{\partial(\rho v_z \phi)}{\partial z} ½dr^2\ d\theta\ dz$$

We can neglect the term in dr^2, so the net rate of flow of scalar Φ out of the CV is

$$\frac{\partial(\rho v_z \phi)}{\partial z} dr\ rd\theta\ dz$$

The sum of the net rates of outflow of scalar Φ is

$$\left[\frac{\partial(\rho v_r \phi)}{\partial r} + \frac{\rho v_r \phi}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta \phi)}{\partial \theta} + \frac{\partial(\rho v_z \phi)}{\partial z}\right] dr\ rd\theta\ dz$$

Finally, we can combine the first and second terms in the brackets into one:

$$\left[\frac{1}{r}\frac{\partial(r\rho v_r \phi)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta \phi)}{\partial \theta} + \frac{\partial(\rho v_z \phi)}{\partial z}\right] dr\ rd\theta\ dz \qquad (2.2)$$

2.3 Diffusion terms

The third term on the right of (1.1) represents the net flow rate of Φ into the CV by diffusion. We shall denote the diffusion flux per unit area by $\boldsymbol{q}$ [kg Φ m^{-2} s^{-1}]. $\boldsymbol{q}$ has components q_r, q_θ, and q_z [kg Φ m^{-2} s^{-1}] in the r, θ and z coordinate directions, respectively. Diffusion is considered positive if it is in the positive coordinate direction.

Referring to Figure 4, the rate of diffusion of Φ through the face perpendicular to the x direction whose centre is P is q_r [kg Φ m^{-2} s^{-1}] times the area of the face, $rd\theta\, dz$ [m^2]; that is,

$$q_r\, rd\theta\, dz \qquad [\mathrm{kg}\,\Phi\,\mathrm{s}^{-1}]$$

The rate of diffusion of Φ through the opposite face whose centre is Q is

$$\left(q_r + \frac{\partial q_r}{\partial r} dr\right)(r + dr)d\theta\, dz \qquad [\mathrm{kg}\,\Phi\,\mathrm{s}^{-1}]$$

and so the net rate of diffusion of Φ *out* of the CV through the faces with centres P and Q is

$$\left(q_r + \frac{\partial q_r}{\partial r} dr\right)(r + dr)d\theta\, dz - q_r\, rd\theta\, dz =$$

$$\frac{\partial q_r}{\partial r} dr\, rd\theta\, dz + \rho v_r q_r\, dr\, d\theta\, dz + \frac{\partial q_r}{\partial r} dr^2\, d\theta\, dz$$

We can neglect the term in dr^2, so the net rate of flow of scalar Φ out of the CV through the faces with centres P and Q is

$$\frac{\partial q_r}{\partial r} dr\, rd\theta\, dz + q_r\, dr\, d\theta\, dz$$

The rate of diffusion of Φ through the face perpendicular to the θ direction whose centre is R is q_θ [kg Φ m^{-2} s^{-1}] times the area of the face, $dr\, dz$; that is,

$$q_\theta\, dr\, dz$$

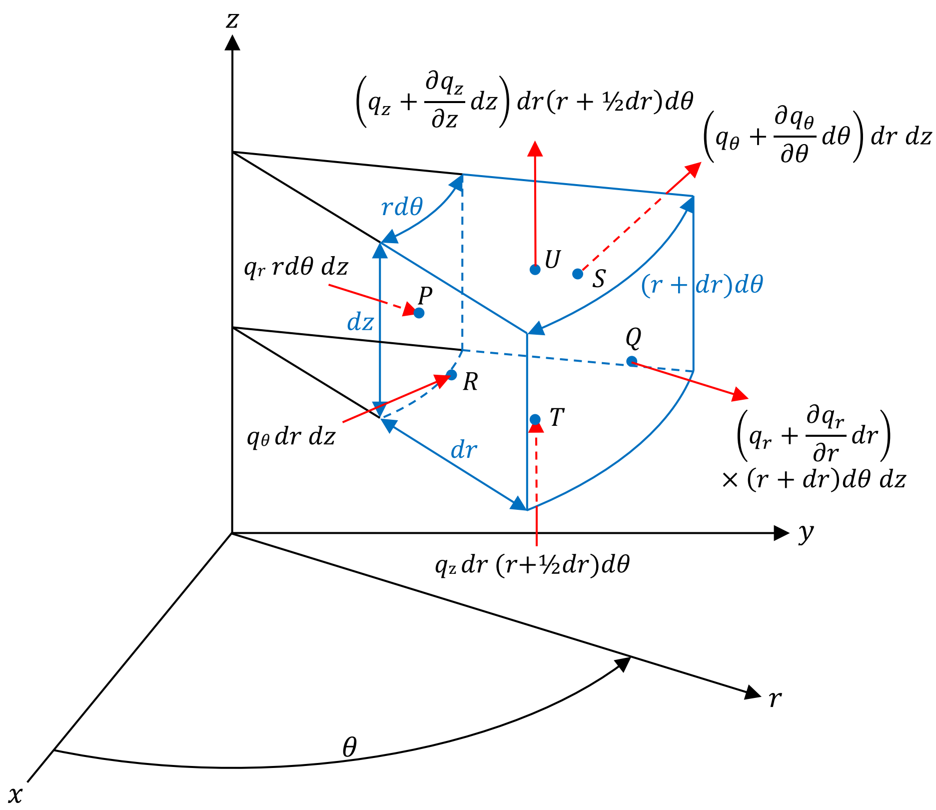
The corresponding rate of diffusion of Φ out of the face with centre S is

$$\left(q_\theta + \frac{\partial q_\theta}{\partial \theta} d\theta\right) dr\, dz$$

so the net rate of diffusion of Φ *out* of the CV through the faces with centres R and S is

$$\frac{\partial q_\theta}{\partial \theta} dr\, d\theta\, dz$$

Figure 4 Diffusion of Φ in the r, θ and z directions



The rate of diffusion of Φ through the face perpendicular to the z direction with centre T is q_z [kg Φ m^{-2} s^{-1}] times the area of the face. The area of the face is

$$dr\,(r + ½dr)d\theta$$

and the rate of diffusion of Φ through the face is

$$q_z\,dr\,(r + ½dr)d\theta$$

The corresponding rate of diffusion of Φ out of the face with centre U is

$$\left(q_z + \frac{\partial q_z}{\partial z}dz\right)dr\,(r + ½dr)d\theta$$

so the net rate of rate of diffusion of Φ out of the CV through the faces with centres T and U is

$$\frac{\partial q_z}{\partial z}dr\,(r + ½dr)d\theta\,dz = \frac{\partial q_z}{\partial z}dr\,rd\theta\,dz + \frac{\partial q_z}{\partial z}½dr^2\,d\theta\,dz$$

We can neglect the term in dr^2, so the net rate of rate of diffusion of Φ out of the CV is

$$\frac{\partial q_z}{\partial z}dr\,rd\theta\,dz$$

The rate of diffusion of Φ *into* the CV, the third term in (1.1), is therefore

$$-\left[\frac{\partial q_r}{\partial r} + \frac{q_r}{r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z}\right]dr\,rd\theta\,dz$$

Finally, we can combine the first and second terms in the brackets into one:

$$-\left[\frac{1}{r}\frac{\partial (rq_r)}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z}\right]dr\,rd\theta\,dz \qquad (2.3)$$

The scalar mass fluxes q_r, q_θ, and q_z [kg Φ m^{-2} s^{-1}] in this equation are related to the scalar property gradients by *Fick's law of diffusion*:

$$q_r = -\rho D\frac{\partial \phi}{\partial r}$$

$$q_\theta = -\frac{\rho D}{r}\frac{\partial \phi}{\partial \theta}$$

$$q_z = -\rho D\frac{\partial \phi}{\partial z}$$

where D [m^2 s^{-1}] is a diffusion coefficient. The net rate of diffusion of Φ into the CV, the third term in (1.1), is therefore

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho D\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\rho D\frac{\partial \phi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\rho D\frac{\partial \phi}{\partial z}\right)\right]dr\,rd\theta\,dz \qquad [\mathrm{kg}\,\Phi\,\mathrm{s}^{-1}] \qquad (2.4)$$

2.4 Source term

The scalar Φ may be created or destroyed as the mixture flows through the CV. If S_Φ [kg Φ kg^{-1} s^{-1}] is the rate at which Φ is created or destroyed per unit mass of mixture then the rate of creation or destruction of Φ in the CV, the fourth term in (1.1), is

$$\rho S_\Phi dr\, rd\theta\, dz \qquad [\mathrm{kg}\ \Phi\ \mathrm{s}^{-1}] \qquad (2.5)$$

2.5 Scalar conservation equation

Substituting the terms (2.2), (2.3), (2.4) and (2.5) into (1.1) and dividing by $dr\ rd\theta\ dz$ gives the conservation equation for the scalar property Φ in cylindrical coordinates:

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r\phi)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta\phi)}{\partial\theta}+\frac{\partial(\rho v_z\phi)}{\partial z}$$

$$=\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho D\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\rho D\frac{\partial\phi}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\rho D\frac{\partial\phi}{\partial z}\right)+\rho S_\Phi \qquad [\ \mathrm{kg}\ \Phi\ \mathrm{m}^{-3}\ \mathrm{s}^{-1}] \qquad (2.6)$$